SUPERINTEGRABLE SYSTEMS ON SPHERE

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Abstract

We consider various generalizations of the Kepler problem to three-dimensional sphere S^3, a compact space of constant curvature. These generalizations include, among other things, addition of a spherical analog of the magnetic monopole (the Poincaré–Appell system) and addition of a more complicated field, which itself is a generalization of the MICZ-system. The mentioned systems are integrable — in fact, superintegrable. The latter is due to the vector integral, which is analogous to the Laplace–Runge–Lenz vector. We offer a classification of the motions and consider a trajectory isomorphism between planar and spatial motions. The presented results can be easily extended to Lobachevsky space L^3.

1 The Kepler problem in $\mathbb{R}^3$

Consider the Kepler problem: a mass point (of unit mass, without losing in generality) moves in the Newtonian field of a fixed center; the intensity of the gravitational interaction γ is constant.

In this problem, in addition to the integral of energy

$$H_0 = \frac{1}{2}\sum_{i=1}^{3} \dot{q}_i^2 + U \tag{1}$$

and the vector integral of angular momentum

$$\boldsymbol{M} = \boldsymbol{q} \times \dot{\boldsymbol{q}}, \tag{2}$$

the equations in three-dimensional Euclidean space $\mathbb{R}^3 = \{q_1, q_2, q_3\}$,

$$\ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad U = -\frac{\gamma}{r}, \quad r^2 = q_1^2 + q_2^2 + q_3^2, \quad \gamma = const, \tag{3}$$

have one more remarkable vector integral, which is due to certain hidden symmetry of the Kepler problem. This vector integral is called the Laplace–Runge–Lenz vector $\boldsymbol{A} = (A_1, A_2, A_3)$. It exists only in the case of Newtonian potential (of all the

central potentials) and can be written as follows:

$$\boldsymbol{A} = \boldsymbol{M} \times \dot{\boldsymbol{q}} + \frac{\gamma}{r}\boldsymbol{q}. \tag{4}$$

Introducing the momenta $\boldsymbol{p} = \dot{\boldsymbol{q}}$, we can rewrite equations (3) and integrals (1), (2), and (4) in the canonical form:

$$\dot{\boldsymbol{p}} = \frac{\partial H_0}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = -\frac{\partial H_0}{\partial \boldsymbol{p}}, \quad \boldsymbol{p},\, \boldsymbol{q} \in \mathbb{R}^3. \tag{5}$$

The Poisson brackets for the components of the integrals $\boldsymbol{M}$ and $\boldsymbol{A}$ are

$$\{M_i, M_j\} = \varepsilon_{ijk}M_k, \quad \{M_i, A_j\} = \varepsilon_{ijk}A_k, \quad \{A_i,\, A_j\} = -2h\varepsilon_{ijk}M_k, \tag{6}$$

where h is the constant of energy (1), $h = \frac{1}{2}\boldsymbol{p}^2 - \frac{\gamma}{r}$, and ε_{ijk} is the Levi–Civita symbol. Depending on the value of h, the algebra of integrals (6) is either $so(4)$ (when $h < 0$) or $so(3,\, 1)$ (when $h > 0$).

Note that, since $(\boldsymbol{M},\, \boldsymbol{A}) = 0$, $\boldsymbol{A}$ is always in the plane of the trajectory. Besides, the vector's direction coincides with the direction of the ellipse's major axis, while its absolute value is proportional to the eccentricity.

The algebra of integrals (6) is an algebra, under which the Kepler problem is invariant. Invariance under a global group of transformations (i. e., for example, under group $SO(4)$ for $h < 0$) was studied by V. A. Fok [9], G. Györgyi [11] and J. Moser [19]. The latter work contains the most general result, which shows that even in the n-dimensional case, the constant energy surface (for $h < 0$) after suitable regularization is topologically equivalent to the bundle of unit vectors tangent to n-dimensional sphere S^n.

Note also that the principal dynamical effect of a redundant algebra of integrals (6) is the fact that the trajectories of system (3) are closed in the configurational and phase spaces.

2 The MICZ-system in $\mathbb{R}^3$. Appell's problem

Consider one more generalization of the Kepler problem, for which an analog of integral (4) exists. To this end, in the phase space $T^*\mathbb{R}^3$ we specify a noncanonical Poisson bracket

$$\{q_i,\, q_j\} = 0, \quad \{q_i,\, p_j\} = \delta_{ij}, \quad \{p_i,\, p_k\} = -\mu\varepsilon_{ijk}\frac{q_k}{r^3} \tag{7}$$

and a Hamiltonian

$$H_1 = \frac{1}{2}\sum_{i=1}^{3} p_i^2 - \frac{\gamma}{r} + \frac{\mu^2}{2r^2}, \quad \gamma,\, \mu = const. \tag{8}$$

Remark. *This system (the differential equations of motion) can as well be obtained with the standard canonical bracket, but in this case the Hamiltonian would contain terms linear in momenta.*

Equations (7), (8) define the MICZ-system (McIntosh- Cisneros-Zwanziger); it describes a particle's motion in the asymptotic field of a self-dual monopole [8]. It was formally studied by Zwanziger [21], McIntosh and Cisneros [18] without any relevant physical interpretation (see also [7]).

Consider some special cases of (7), (8). The Kepler problem can be obtained if we put $\mu = 0$. Putting $\gamma = 0$ and $\mu = 0$ in the Hamiltonian (8), not in the bracket (7), we have the classical integrable Poincaré problem of a particle moving in the field of a magnetic monopole. As it was shown by Poincaré, the particle's trajectories in this case are geodesics of a circular cone.

P. Appell considered a more general problem of a particle moving in the field of a Newtonian center and in the field of a magnetic monopole, assuming that the center and the monopole coincide [1]. This problem occurs if $\mu = 0$ in (8) (in the bracket (7), however, $\mu \neq 0$). In this case, the trajectory is a conic section in the involute of the circular cone, while the integral of areas is preserved during the motion. On the cone itself, the trajectories are, generally, not closed.

There is no analog of the integral $\boldsymbol{A}$ (4) in the Poincaré and Appell problems, but it exists for the system (7), (8). As for the vector integral of angular momentum $\boldsymbol{M}$, it exists for all the above problems. Indeed [3], the vector functions

$$\boldsymbol{M} = \boldsymbol{q} \times \boldsymbol{p} + \mu \frac{\boldsymbol{q}}{r}, \tag{9}$$

$$\boldsymbol{A} = \frac{1}{\sqrt{|2H_1|}} \left(\boldsymbol{p} \times \boldsymbol{M} - \frac{\boldsymbol{q}}{r} \right) \tag{10}$$

form the algebra of integrals of (7), (8), which is isomorphic to $so(4)$ for $H_1 < 0$ and to $so(3,1)$ for $H_1 > 0$.

Again the trajectories are conic sections, and, since $(\boldsymbol{M}, \boldsymbol{q}/r) = -\mu$, belong to the circular cone with cone angle $\theta = \arccos \mu/|\boldsymbol{M}|$ and the axis of symmetry, defined by $\boldsymbol{M}$.

Various generalizations of the Laplace–Runge–Lenz integrals to dynamical systems in the Euclidean space were studied in [16].

3 The Kepler problem on three-dimensional sphere S^3 (and on Lobachevsky space L^3)

Consider analogs of the Kepler problem in some simple non-Euclidean spaces of constant curvature — three-dimensional sphere S^3 and Lobachevsky space. We will

discuss the spherical case in more detail. However, all the results, after appropriate revision, can be extended to Lobachevsky space.

Let the three-dimensional sphere S^3 be embedded into the four-dimensional Euclidean space $\mathbb{R}^4 = \{q_0,\, q_1,\, q_2,\, q_3\}$ and given by equation

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = R^2, \tag{11}$$

where R is the sphere's radius.

Introduce spherical coordinates on S^3:

$$\begin{gathered} q_0 = R\cos\theta, \quad q_1 = R\sin\theta\cos\varphi, \\ q_2 = R\sin\theta\sin\varphi\cos\psi, \quad q_3 = R\sin\theta\sin\varphi\sin\psi. \end{gathered} \tag{12}$$

Consider the motion of a particle in the field of a Newtonian center, placed at one of the poles, $\theta = 0$, of the three-dimensional sphere. It is well known [17, 12, 10, 13, 15, 20] that the analog of Newtonian potential on sphere is

$$U = -\gamma\cot\theta = -\gamma\frac{q_0}{|\boldsymbol{q}|}, \quad \boldsymbol{q}^2 = q_1^2 + q_2^2 + q_3^2, \quad \boldsymbol{q} = (q_1,\, q_2,\, q_3). \tag{13}$$

Recall that the potential (13) can be obtained either by solving the Laplace–Beltrami equation on sphere S^3 (see below, (29)). This equation is invariant under group $SO(3)$ and has a singularity at the pole $\theta = 0$, or by extending Bertrand's theorem to sphere [14, 17].

In independent coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$, the Lagrangian of the problem in question is

$$L = \frac{1}{2}(\dot{\boldsymbol{q}}^2 + q_0^{-2}(\boldsymbol{q}, \dot{\boldsymbol{q}})^2)^2 - U(\boldsymbol{q}), \tag{14}$$

where q_0 is found from (11), namely, $q_0 = \pm\sqrt{R^2 - \boldsymbol{q}^2}$. After introducing the momenta

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \dot{\boldsymbol{q}} + \frac{(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\sqrt{R^2 - \boldsymbol{q}^2}}\boldsymbol{q}, \tag{15}$$

the equations of motion can be represented in the canonical Hamiltonian form with Hamiltonian

$$H = \frac{1}{2}\boldsymbol{p}^2 - \frac{1}{2R^2}(\boldsymbol{p},\, \boldsymbol{q})^2 + V(\boldsymbol{q}). \tag{16}$$

These equations have a vector integral of angular momentum

$$\boldsymbol{M} = \boldsymbol{p} \times \boldsymbol{q} = \dot{\boldsymbol{q}} \times \boldsymbol{q} \tag{17}$$

(it exists as well for every "central" potential V that depends only on $|\boldsymbol{q}| = \sqrt{q_1^2 + q_2 + q_3^2}$) and an analog of the Laplace–Runge–Lenz integral

$$\boldsymbol{A} = q_0\boldsymbol{p} \times \boldsymbol{M} + \gamma R^2\frac{\boldsymbol{q}}{|\boldsymbol{q}|}. \tag{18}$$

The components of M_i and R_i commute in the following way:

$$\begin{gathered}\{M_i,\, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_i,\, A_j\} = -\varepsilon_{ijk} A_k, \\ \{A_i\; A_j\} = 2(R^2 h - \boldsymbol{M}^2)\varepsilon_{ijk} M_k.\end{gathered} \tag{19}$$

(This algebra was discussed in several papers [12, 10].)

The Casimir functions of the nonlinear Poisson structure (19) are

$$F_1 = (\boldsymbol{M}, \boldsymbol{A}), \quad F_2 = \boldsymbol{A}^2 - \frac{2h}{\lambda}\boldsymbol{M}^2 + (\boldsymbol{M}^2)^2, \tag{20}$$

while its symplectic leaf is four-dimensional (i. e., the rank of (19) is four). Here, $\lambda = 1/R^2$ is the curvature of the space. For real motions, the Kepler problem gives

$$F_1 = 0, \quad F_2 = \gamma^2 R^4. \tag{21}$$

The compactness of the symplectic leaf (21) is defined by the curvature of space, λ, and the value of the constant of energy:

1. When $\lambda = 0$: $h < 0$ — compact, $h \geq 0$ — noncompact.
2. When $\lambda > 0$: always compact.
3. When $\lambda < 0$: $h < 0$ and $h^2 > \gamma^2$ — the leaf (21) is disconnected, one component is compact, while the other is noncompact; $h > -\gamma$ — the leaf is connected, but noncompact.

The trajectories of the Kepler problem on sphere (and pseudosphere) are conic sections, the generalization of Kepler's laws to this case was done in [14, 13, 5]. In the paper [6], bifurcational analysis of the Kepler problem on S^3 and L^3 was performed, and the action-angle variables were introduced (see also [4]).

4 Generalization of the Poincaré and Appell problems to S^3

First, we obtain the Hamiltonian form of the equations of a particle's motion on three-dimensional sphere S^3 under generalized potential forces. Indeed, consider the Lagrangian

$$L = \frac{1}{2}\big(\dot{\boldsymbol{q}} + q_0^{-2}(\boldsymbol{q}, \dot{\boldsymbol{q}})^2\big) - \big(\dot{\boldsymbol{q}}, \boldsymbol{W}(\boldsymbol{q})\big) - U(\boldsymbol{q}), \tag{22}$$

where $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{q}) = (W_1,\, W_2,\, W_3)$ is the vector potential. Introducing the generalized momenta

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \dot{\boldsymbol{q}} - \boldsymbol{W} + \frac{(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\sqrt{R^2 - \boldsymbol{q}^2}}\boldsymbol{q}, \tag{23}$$

we obtain the Hamiltonian

$$H = \frac{1}{2}(\boldsymbol{p} + \boldsymbol{W})^2 - \frac{1}{2R^2}(\boldsymbol{p} + \boldsymbol{W}, \boldsymbol{q})^2 + U(\boldsymbol{q}) \tag{24}$$

and the canonical Poisson bracket ($\{q_i, p_j\} = \delta_{ij}$). Due to a number of considerations, it is more convenient to study Hamiltonian equations written in terms of slightly modified momenta $\tilde{\boldsymbol{p}} = \boldsymbol{p} + \boldsymbol{W}$, which form the following noncanonical Poisson brackets

$$\begin{gathered}\{\tilde{p}_i, \tilde{p}_j\} = \frac{\partial W_i}{\partial q_j} - \frac{\partial W_j}{\partial q_i} = B_{ij}, \\ \{q_i, \tilde{p}_j\} = \delta_{ij}, \quad \{q_i, q_j\} = 0,\end{gathered} \tag{25}$$

where $\boldsymbol{B} = \operatorname{rot} \boldsymbol{W}$. The Hamiltonian (24) simplifies, in this case, to:

$$H = \frac{1}{2}\tilde{\boldsymbol{p}}^2 - \frac{1}{R^2}(\tilde{\boldsymbol{p}}, \boldsymbol{q})^2 + U(\boldsymbol{q}). \tag{26}$$

In the case of three-dimensional sphere, an analog of the vector potential of a magnetic monopole can be obtained in the following way.

Electromagnetic field tensor in vacuum satisfies the Maxwell equations

$$\begin{gathered}\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad \alpha, \beta = 0, 1, 2, 3 \\ \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}F^{\alpha\beta}) = 0 \quad \left(\partial_\alpha = \frac{\partial}{\partial x_\alpha}\right),\end{gathered} \tag{27}$$

where $\|g_{\alpha\beta}\|$ is the metric of space-time, $g = \det \|g_{\alpha\beta}\|$.

For S^3, the metric of space-time in the spherical coordinates (12) is

$$dS^2 = c^2 dt^2 - R^2\big(d\theta^2 + \sin^2\theta(d\varphi^2 + \sin^2\varphi d\psi^2)\big). \tag{28}$$

Let i, j, k stand only for the spatial indices, while g_* denotes the spatial portion of the metric, taken with the negative sign. We will search for the solution, similar to that for a magnetic monopole in flat space, in the form

$$F_{0i} = 0, \quad \sqrt{g_*}F^{ij} = \varepsilon^{ijk}\partial_k f.$$

From (27) we find the equation for the unknown function f

$$\partial_k\left(\sqrt{g_*}g^{ik}\partial_i f\right) = 0,$$

coinciding with the Laplace–Beltrami equation. The solution, invariant under group $SO(3)$ (i. e. independent of ψ, φ), satisfies the equation

$$\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\frac{\partial f}{\partial\theta}\right) = 0 \tag{29}$$

and looks as follows:

$$f = \alpha \cot\theta, \quad \alpha = const. \tag{30}$$

Remark. *For Lobachevsky space L^3, a similar reasoning yields $f = \alpha \coth\theta$, $\alpha = const$.*

The vector potential of the magnetic monopole $\boldsymbol{W}$ is found from $F_{ij} = \partial_i W_j - \partial_j W_i$. In terms of spherical coordinates (θ, φ, ψ), it reads:

$$W_\theta = 0, \quad W_\varphi = 0, \quad W_\psi = \alpha R \cos\varphi.$$

In terms of variables $q_0, \boldsymbol{q}$, it can be written as

$$\boldsymbol{W} = \left(0,\, \alpha \frac{q_1}{|\boldsymbol{q}|}\frac{q_3}{q_2^2+q_3^2},\, -\alpha\frac{q_1}{|\boldsymbol{q}|}\frac{q_2}{q_2^2+q_3^2}\right), \tag{31}$$

while for $\boldsymbol{B} = \operatorname{rot} \boldsymbol{W}$, we have

$$\boldsymbol{B} = -\frac{\alpha}{|\boldsymbol{q}|^3}\boldsymbol{q}. \tag{32}$$

Consider a particle, moving on S^3 in the field of a Newtonian center and in the field of a magnetic monopole, the center and the monopole being placed at the pole $\theta = 0$. This is a spherical analog of the Appell problem. The Hamiltonian of the problem is either (24) or (26) with $\boldsymbol{W}(\boldsymbol{q})$ and $U(\boldsymbol{q})$ defined, respectively, by (31) and (13). Hamiltonian equations always admit the integral of angular momentum

$$\boldsymbol{M} = \tilde{\boldsymbol{p}} \times \boldsymbol{q} - \alpha\frac{\boldsymbol{q}}{|\boldsymbol{q}|} = \dot{\boldsymbol{q}} \times \boldsymbol{q} - \alpha\frac{\boldsymbol{q}}{|\boldsymbol{q}|}. \tag{33}$$

To simplify the reasoning, we put $R = 1$ and write the Lagrangian in the spherical coordinates (12)

$$L = \frac{1}{2}\left(\dot\theta^2 + \sin^2\theta\dot\varphi^2 + \sin^2\theta\sin^2\varphi\dot\psi^2\right) + \alpha\cos\varphi\dot\psi - U(\theta). \tag{34}$$

Suppose that the angular momentum vector (33) is aligned with the axis q_1 in the space q_1, q_2, q_3; then

$$M_2 = M_3 = 0, \quad \dot\varphi = 0, \quad \sin^2\theta\dot\psi = \frac{\alpha}{\cos\varphi_0} = const. \tag{35}$$

The latter relation is a generalization of Kepler's second law. Consider the invariant surface in S^3 given by $\left(\boldsymbol{M},\, \frac{\boldsymbol{q}}{|q|}\right) = const$. On this surface, choose a point, which is 2θ away from the particle. Then, the great-circle arc, joining the origin of coordinates with the chosen point sweeps equal areas in equal time intervals. (Indeed, the time rate of change of the area is $\frac{dS}{dt} = \left(\int_0^{2\theta}\sin\sigma\, d\sigma\right)\frac{d\psi}{dt} = 2\sin^2\theta\dot\psi$.)

Taking into account the integral of energy $E = h$ and (35), we obtain

$$\dot{\theta} = \sqrt{2(h - U_c(\theta))}, \quad U_c(\theta) = U(\theta) + \frac{1}{2}\frac{\alpha^2\theta n^2\varphi_0}{\sin^2\theta} \tag{36}$$

and the explicit expression for the trajectory in terms of quadratures:

$$\frac{\alpha\, d\theta}{\sin^2\theta\sqrt{2(\tilde{h} - \tilde{U}) - \frac{\alpha^2\sin^2\varphi_0}{\sin^2\theta}}} = d\psi, \tag{37}$$

where $\tilde{h} = h\cos^2\varphi_0^2$, $\tilde{U} = U\cos^2\varphi_0$.

When $\gamma = 0$, an explicit expression in terms of quadratures for the analog of the Poincaré problem is obtained from (37), and the trajectories are geodesics on the invariant cone defined by $(\boldsymbol{M},\, \boldsymbol{q}/|\boldsymbol{q}|) = const$. As it was noted above, in the case of the Appell problem's analog the trajectories are conic sections (i. e. ellipses, hyperbolas, parabolas) on the plane development of the cone. Generally, these are not closed on the cone, but there is a possibility to "adjust" the potential (13) so that the trajectories will be always closed in the presence of a monopole. Due to this "tuning", the Euclidean MICZ-model can be generalized to the spherical case, for which the Laplace–Runge–Lenz integral exists.

5 Generalized MICZ-model

Consider a motion in the field of a monopole and in the field with the potential

$$U(\theta) = -\gamma\cot\theta + \frac{1}{2}\frac{\mu}{\sin^2\theta}, \quad \gamma, \mu = const. \tag{38}$$

The trajectory in this case is given by (37). If

$$\mu = \alpha^2, \tag{39}$$

the trajectory is

$$\frac{\alpha d\theta}{\sin^2\theta\sqrt{2\tilde{h} + 2\tilde{\gamma}\cot\theta - \frac{\alpha^2}{\sin^2\theta}}} = d\psi. \tag{40}$$

The trajectory (40) is closed, and the gnomonic projection gives us the conic section

$$\tan\theta = \frac{p}{1 + e\cos(\psi - \psi_0)} \tag{41}$$

with the following focal parameter and eccentricity:

$$p = \frac{\alpha^2}{\gamma\cos^2\varphi_0}, \quad e = \sqrt{1 + \frac{2\alpha^2}{\gamma^2\cos^2\varphi_0}\left(h - \frac{\alpha^2}{2\cos^2\varphi_0}\right)}.$$

The expression in terms of quadratures for $\dot{\theta}$ (36) is

$$\begin{aligned} \dot{\theta}^2 &= 2h + 2\gamma \cot\theta - \frac{\alpha^2}{\cos^2\varphi_0 \cdot \sin^2\theta} = \\ &= 2h + 2\gamma\cot\theta - \frac{c^2}{\sin^2\theta} = f(\theta,\, c,\, h), \end{aligned} \tag{42}$$

where $c = \alpha^2/\cos^2\varphi_0$. Let us plot the bifurcational diagram of the problem's solutions on the parameter plane $(c^2,\, h)$. To this end, recall that at critical points $(c_*,\, h_*)$ on a bifurcation curve, $f(\theta_0,\, c_*,\, h_*) = f_0'(\theta_0,\, c_*,\, h_*) = 0$. As a result, we have two curves (Fig. 1):

$$\text{I. } 2h = c^2 - \frac{\gamma^2}{c^2}; \quad \text{II. } c^2 = 0.$$

Besides, since $c = \alpha^2/\cos^2\varphi_0$, the inequality $c^2 > \alpha^4$ also holds. Thus, in the plane defined by the constants h, c^2 (see Fig. 1), the domain of allowable values h, c^2 lies above the line $c^2 = \alpha^4$ and below the hyperbola defined by I. For a point of this plane above $h = \frac{1}{2}c^2$, the particle moves only in the upper half-plane of the sphere. Otherwise, the particle can also move into the other half-plane.

It is easy to formulate an analog of Kepler's third law, coinciding with the traditional law for a curved space [13]. Indeed, since $\sin^2\theta\dot{\psi} = c$, we have

$$dt = \frac{\sin^2\theta d\psi}{c}.$$

Therefore,

$$\begin{aligned} T &= \frac{1}{c}\int_0^{2\pi} \sin^2\theta(\psi)d\psi = \frac{p^2}{c}\int_{-\pi}^{\pi} \frac{d\psi}{p^2 + (1 + e\cos\psi)^2} = \\ &= \frac{\pi}{\sqrt{\gamma}}\sqrt{\frac{h}{\gamma} + \sqrt{1 + \frac{h^2}{\gamma^2}}} \cdot \left(1 + \frac{h^2}{\gamma^2}\right)^{-1/2}. \end{aligned} \tag{43}$$

This dependence of the orbital period on energy can be easily transformed into the dependence on (angular) length of the semi-major axis

$$T = \frac{\pi}{\sqrt{\gamma}}\sqrt{-\tan a + \sqrt{1 + \tan^2 a}}(1 + \tan^2 a)^{-1/2}, \tag{44}$$

where $\tan a = -\frac{\gamma}{h}$.

As in the Kepler problem (on $\mathbb{R}$ and S^2), closedness of the trajectories is closely connected with some hidden symmetry of the problem, i. e., with existence of a vector integral of the Laplace–Runge–Lenz type.

For the system (38), (39) this vector can be written as

$$\boldsymbol{A} = q_0 \tilde{\boldsymbol{p}} \times \boldsymbol{M} + \gamma R^2 \frac{\boldsymbol{q}}{|\boldsymbol{q}|}. \tag{45}$$

The Poisson brackets for the components of $\boldsymbol{A}$ and the components of the integral of angular momentum (33) are:

$$\begin{gathered} \{M_i,\, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_i,\, R_j\} = -\varepsilon_{ijk} R_k, \\ \{R_i,\, R_j\} = 2\varepsilon_{ijk} \left(R^2 h - \boldsymbol{M}^2 + \frac{1}{2}\alpha^2 \right) M_k. \end{gathered} \tag{46}$$

As before, we can specify the conditions (in terms of the curvature of the space and the value of the integral of energy), under which the symplectic leaf (23) is compact.

6 Trajectory isomorphism for central potential systems on S^2 and $\mathbb{R}^2$

For $U = U(r)$, the equations (3) define a central potential system on $\mathbb{R}^3$. If $U = U(\theta)$ in the Lagrangian (14), then we have a central potential system on S^3. These systems, respectively, have flat ($\mathbb{R}^2$) and spherical (S^2) invariant manifolds. These two-dimensional systems can be shown to be related, using the central (gnomonic) projection (from the center of the sphere tangent to the plane at the attracting center) and some suitable change of time.

Following Serret and Appell [20, 2], consider a system in $\mathbb{R}^2$ with the following equations of motion (in polar coordinates):

$$\frac{d}{dt}\left(\frac{\partial T_p}{\partial \dot{\rho}}\right) = \mathrm{R}; \quad \frac{d}{dt}\left(\frac{\partial T_p}{\partial \dot{\varphi}}\right) = \Phi; \tag{47}$$

where T_p is the kinetic energy of a point in the plane,

$$T_p = \frac{1}{2}\left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2\right), \tag{48}$$

while R, Φ stand for certain generalized forces (generally, non-potential).

Let us perform the transformation of coordinates (the gnomonic projection from Fig. 2), forces and time:

$$\begin{gathered} \rho = \tan\theta, \quad \varphi = \psi, \quad dt = \cos^{-2}\theta d\tau \\ R = \cos^2\theta\, \Theta, \quad \Phi = \cos^2\theta\, \Psi. \end{gathered} \tag{49}$$

This results in a system on S^2:

$$\frac{d}{d\tau}\left(\frac{\partial T_s}{\partial \theta'}\right) = \Theta, \quad \frac{d}{d\tau}\left(\frac{\partial T_s}{\partial \psi'}\right) = \Psi, \tag{50}$$

where $\theta' = \frac{dT_s}{d\tau}$, $\psi' = \frac{d\psi}{d\tau}$, while T_s is the kinetic energy of a point on the sphere,

$$T_s = \frac{1}{2}\left(\theta' + \sin^2\theta\psi'^2\right). \tag{51}$$

It is easy to see that

Statement. *There exists a trajectory isomorphism between the Lagrangian system in $\mathbb{R}^2$, with central potential*

$$L = \frac{1}{2}(\dot\rho^2 + \rho^2\dot\psi^2) + U(\rho),$$

and the Lagrangian system on S^2, with central potential of the form

$$L = \frac{1}{2}(\dot\theta^2 + \sin^2\theta\dot\psi^2) + U(\tan\theta).$$

To prove that, it is sufficient to put in (47)–(51)

$$\Phi = \Psi = 0$$

$$\mathrm{R} = -\frac{\partial U}{\partial \rho}, \quad \Theta = -\frac{\partial U}{\partial \theta} = -\frac{\partial U}{\partial \rho}\cdot\frac{\partial \rho}{\partial \theta} = \frac{\mathrm{R}}{\cos^2\theta}.$$

These transformations easily bring the plane Kepler problem to its analog on sphere.

Note that under the transformation (49) a potential field of forces can be transformed into a non-potential field, and vice versa.

If we consider the inverse transformation (49) as a transformation from sphere to plane, we have to adopt negative values of ρ. In this case, ρ is negative when the trajectory on the sphere crosses its equator, or, in the plane (ρ, φ), when the trajectory jumps from $+\infty$ to $-\infty$. If, instead of $\rho = \tan\theta$, we consider $\xi = \cot\theta$, then the trajectory in the plane (ξ, φ) is continuous.

It is also easy to show that the described isomorphism can be extended to the generalized potential systems discussed in Sections 4, 5. Note also that the transformation (49), applied to the Kepler problem on S^2, can be used to generalize the Bohlin (Levi–Civita) regularization. It can be shown that *on a fixed energy level, the Kepler problem is reduced to the harmonic oscillator problem.*

Another interesting property of the Kepler system in $\mathbb{R}^2$ (due to Hamilton) is that the velocity hodograph for a moving point is a circle with a displaced center. A similar vector can be specified for the Kepler problem on S^2:

$$\boldsymbol{\pi} = \frac{\dot{\boldsymbol{x}}}{1 + \dot{\boldsymbol{x}}^{\mathbf{2}}/R^2},$$

where $\boldsymbol{x} = (\rho \cos \varphi, \rho \sin \varphi)$ is the radius-vector of a point under the gnomonic projection (49). The $\boldsymbol{\pi}$ hodograph can be easily shown to be a circle with a displaced center.

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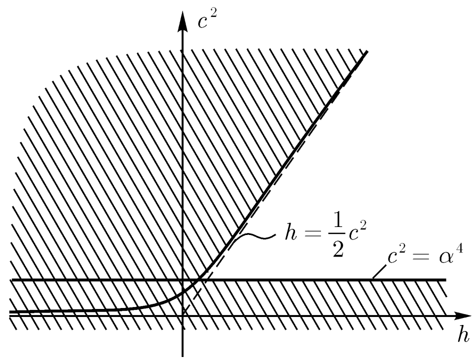

Figure 1:

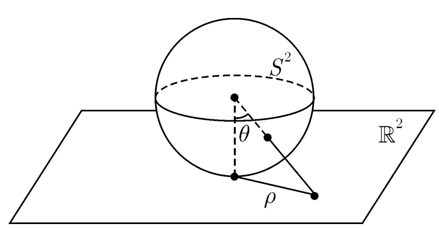

Figure 2: The gnomonic projection